

Hierarchical Spherical Model as a Viscosity Limit of Corresponding $O(N)$ Heisenberg Model

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Abstract

The $O(N)$ Heisenberg and spherical models with interaction given by the long range hierarchical Laplacean are investigated. Two classical results are adapted. The Kac–Thompson solution [KT] of the spherical model, which holds for spacially homogeneous interaction, is firstly extended to hierarchical model whose interaction fails to be translation invariant. Then, the convergence proof of $O(N)$ Heisenberg to the spherical model by Kunz and Zumbach [KZ] is extended to the long range hierarchical interaction. We also examine whether these results can be carried over as the size of the hierarchical block L goes to 1. These extensions are considered a preliminary study prior the investigation of the model by renormalization group given in [MCG] where central limit theorems for the spherical ($N = \infty$) model on the local potential approximation ($L \downarrow 1$) are then established from an explicit solution of the associate nonlinear first order partial differential equation.

1 Introduction

Motivations. In the present work we initiate a geometric study of partial differential equations related to the renormalization group transformation (RGT) of a d -dimensional N -component hierarchical spin system in the limit as the block size L^d goes to 1.

For one-component hierarchical spin system, the evolution equation corresponding to $L \downarrow 1$ limit, known as the local potential approximation, began to be investigated by Felder [F] who has constructed, in that seminal work, a family of global stationary solutions – nontrivial fixed point analogs of the corresponding d -dimensional models with $d > 2$ and L^d an integer > 1 . Except by a preliminary study and numerical simulation of the corresponding N -component spin equation by Zumbach [Z] and a refinement of Felder’s approach by Lima [L], no much attention has been paid to the evolution equation model.

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Very recently, the renormalization group trajectories of $d = 4$ dimensional hierarchical $O(N)$ spin model with block size $L^d = 2$ has been investigated by Watanabe [W] (for previous investigations, see [GK, K] and references therein). Starting from the uniform “a priori” measure supported in the N -dimensional sphere of radius \sqrt{N} , the critical trajectory has shown to converge to the Gaussian fixed point for sufficiently large N . To control such trajectory, which starts far away from the fixed point, the exactly solved $O(\infty)$ trajectory has been used together with two key ingredients: reflection positivity and the Lee–Yang property of single-site “a priori” measures. The former ingredient gives uniform convergence of $O(N)$ trajectories to $O(\infty)$ trajectories. The latter property has been previously employed by Kozitsky [K] to establish central limit theorems for the hierarchical $O(N)$ spin model with block size $L^d \geq 2$ and $d > 4$ (in his notation $L^d = \delta$ and $2/d = \lambda \in (0, 1/2)$) at the critical inverse temperature β_c and below. Watanabe’s analysis, based in his joint work with Hara and Hattori [HHW] on the critical trajectory for the hierarchical Ising model ($N = 1$), in contradistinction, deals with the borderline $d = 4$ case and does not require closeness to the fixed point.

Although the analysis of the RGT with $L^d \geq 2$ fixed is expected to be simplified considerably in the $L \downarrow 1$ limit, none of the results on the critical trajectories can be carried to the limit as the above mentioned ingredients do not hold if L^d is not an integer. To establish weak convergence of the hierarchical $O(N)$ Heisenberg equilibrium measure to the corresponding spherical equilibrium measure as $N \rightarrow \infty$ in the local potential approximation ($L \downarrow 1$) an entirely new method of analysis has to be developed from scratch.

The present investigation establishes some classical results on the $O(N)$ Heisenberg and spherical model with short range discrete Laplacean interaction replaced by the long range hierarchical Laplacean and examines whether they can be carried over as L goes to 1. Kac–Thompson solution of the spherical model [KT], which holds for spatially homogeneous interaction, is presented in Section 2 and extended in Section 3 to hierarchical model whose interaction fails to be translation invariant. Kac–Thompson’s asymptotic analysis is then applied to the moment generating function of the block spin random variable and normal fluctuations are established for the spherical model for $\beta < \beta_c$. To our knowledge, such application and consequences are new. In Section 4 the convergence proof of $O(N)$ Heisenberg to the spherical model by Kunz and Zumbach [KZ] is extended to the long range hierarchical interaction. The prove holds for the free energy and the moments generating function. These extensions are considered a preliminary study prior the investigation of the model by renormalization group. In a subsequent work [MCG] we establish central limit theorems for the spherical model on the local potential approximation from an explicit solution of the associate nonlinear first order partial differential equation given by (1.7) with $N = \infty$.

Viscosity Limit. The hierarchical Heisenberg model on a box $\Lambda \subset \mathbb{Z}^d$ of size $n = L^{dK}$ is defined by an $O(N)$ invariant equilibrium measure

$$d\nu_n^{(N)}(\mathbf{x}) = \frac{1}{Z_n} \exp \left\{ -\frac{1}{2} (\mathbf{x}, A\mathbf{x}) \right\} \prod_{j=1}^n d\sigma_0^{(N)}(x_j) \quad (1.1)$$

where $\mathbf{x} = (x_1, \dots, x_n)$ denotes an element of the configuration space $\Omega_n = \mathbb{R}^N \times \dots \times \mathbb{R}^N$, $A = J \otimes I$ the tensor product of the $n \times n$ coupling hierarchical matrix J (see (3.2)) with the $N \times N$ identity matrix I and $\sigma_0(x)$ the a priori uniform measure on the N -dimensional sphere $|x|^2 = \beta N$ of radius $\sqrt{\beta N}$.

The invariance of J under the block spin transformation (3.1) allows to describe the laws of

equilibrium studying the dynamics of a recursion relation

$$\sigma_k(x) = \mathcal{R}\sigma_{k-1}(x)$$

in the space of single-site “a priori” measures, which in terms of their characteristic functions

$$\phi_k^{(N)}(z) = \int d\sigma_k^{(N)}(x) \exp(iz \cdot x) , \quad (1.2)$$

reads

$$\phi_k^{(N)}(z) = \frac{1}{N_k} \exp\left(\frac{-1}{2}\Delta\right) \left(\phi_{k-1}^{(N)}(L^{-\gamma/2}z)\right)^{L^d} \quad (1.3)$$

with $\gamma = d + 2$ and initial condition

$$\phi_0^{(N)}(z) = \frac{J_{N/2-1}(\sqrt{\beta N}|z|)}{(\sqrt{\beta N}|z|/2)^{N/2-1}} \Gamma(N/2) = \varphi_0^{(N)}(|z|) \quad (1.4)$$

Here, $\exp(t\Delta)$ is the semi-group generated by the N -dimensional Laplacean $\Delta = \partial^2/\partial z_1^2 + \dots + \partial^2/\partial z_N^2$, N_k is chosen so that $\phi_k(0) = 1$ holds for all $k = 1, \dots, K$ and $J_\alpha(x)$ is the Bessel function of order α . Note that $\phi_k^{(N)}(z) = \varphi_k^{(N)}(|z|)$ depends only on the norm $|z|^2 = z \cdot z$ and β is the inverse temperature.

We shall now explain our title. Since $\psi(t, z) = \exp(t\Delta) \psi_0(z) = \exp(-U(t, z))$ satisfies the heat equation with initial condition $\psi(0, \cdot) = \psi_0$, U satisfies $U_t - \Delta U + |\nabla U|^2 = 0$ with $U(0, \cdot) = U_0 = -\ln \psi_0$ and the sequence $u_k^{(N)}(x)$, $k = 0, 1, \dots$, defined (with $|z|^2 = -Nx$) by

$$\varphi_k^{(N)}(\sqrt{-Nx}) = \exp\left(-Nu_k^{(N)}(x)\right)$$

can be obtained by solving a nonlinear heat equation¹

$$u_t - \frac{2}{N}xu_{xx} - u_x + 2xu_x^2 = 0 \quad (1.5)$$

up to time $t = 1/2$ starting from the initial conditions $u(0, x) = L^d u_{k-1}^{(N)}(L^{-\gamma}x)$:

$$u_k^{(N)}(x) = u(1/2, x) - u(1/2, 0) \quad (1.6)$$

(the solution at $x = 0$ is subtracted to satisfy $\varphi_k^{(N)}(0) = e^{-Nu_k^{(N)}(0)} = 1$). This is the starting point of Watanabe’s investigation [W] (see also [HHW]) on Taylor coefficients $\nu_{2l,k}^{(N)}$, $l \geq 1$, of the sequence of functions $v_k^{(N)}(\zeta) = -\ln \varphi_k^{(N)}(\sqrt{N}\zeta)/N$ around $\zeta = 0$. As already recognized by Watanabe, the parabolic equation (1.5) becomes a first order hyperbolic equation when $N \rightarrow \infty$ and the expansion in powers of $1/N$ of a trajectory $\left(u_k^{(N)}\right)_{k \geq 0}$ for the N -vectorial hierarchical model is a singular perturbation about the corresponding trajectory for the spherical hierarchical model.

The local potential approximation replaces the exponent $1/2$ in (1.3) by $(L - 1)/2$ and the interval of time evolved by (1.5) tends to 0 when $L \downarrow 1$. As a consequence, defining $u^{(N)}(t, x) =$

¹Subindex t and x refer to partial derivatives with respect to independent variables.

$u_k^{(N)}(x)$ for $t = k \ln L$ and taking the limit $L \downarrow 1$, $k \rightarrow \infty$ with t fixed, the recursive initial value problem (1.5) and (1.6) for $\left(u_k^{(N)}\right)_{k \geq 0}$ turn into a genuine initial value problem given by

$$u_t^{(N)} - \frac{2}{N} x u_{xx}^{(N)} - u_x^{(N)} + 2x (u_x^{(N)})^2 + \gamma x u_x^{(N)} - du^{(N)} + u_x^{(N)}(t, 0) = 0 \quad (1.7)$$

with $u^{(N)}(0, x) = -\ln \varphi_0^{(N)}(\sqrt{-Nx})/N \equiv u_0^{(N)}$. Comparing to (1.5), (1.7) includes three extra terms, the last one ensures $u^{(N)}(t, 0) = 0$ for all $t \geq 0$, corresponding to the operations of dilation, multiplication and normalization performed between two consecutive evolutions of (1.5). Note that the stationary solution $u^*(x) = -x$ of (1.7) corresponds to the Gaussian fixed point of (1.3). In the second work of our series [MCG] we give a geometric description of the trajectory $\{u^{(\infty)}(t, x), t \geq 0\}$, in the viscosity limit $N = \infty$,² at and above the critical inverse temperature. Our third investigation will address the solution $\{u^{(N)}(t, x), t \geq 0\}$ of (1.7) as a (singular) perturbation about the critical trajectory $\{u^{(\infty)}(t, x), t \geq 0\}$.

Statement of Results. Equilibrium laws of the model are described by the distribution of block spin random variable $X_n^\gamma = n^{-\gamma/(2d)} \sum_{j=1}^n x_j$ in the limit as $n \rightarrow \infty$ with a properly chosen γ . The characteristic function with respect to the equilibrium measure (1.1) of the block variable with $\gamma = d + 2$ reads

$$\begin{aligned} \Phi_n^{(N)}(z) &= \int \exp \left(i L^{-K(d+2)/2} \sum_{j=1}^n x_j \cdot z \right) d\nu_n^{(N)}(\mathbf{x}) \\ &= \int \exp(i x \cdot z) d\sigma_K^{(N)}(x) = \varphi_K^{(N)}(|z|). \end{aligned}$$

The equilibrium distribution $\sigma_K^{(N)}(x) = \nu_n^{(N)}(X_n^\gamma \leq x)$ converges weakly in the thermodynamic limit $n = L^{dK} \rightarrow \infty$ to $\sigma^{(N)}(x) = \nu^{(N)}(X^\gamma \leq x)$ if $\varphi_K^{(N)}(|z|)$ is continuous at origin and converges pointwise to a continuous (at origin) function $\varphi^{(N)}(|z|)$. Hence, the equilibrium distribution $\nu^{(N)}(X^\gamma \leq x)$ converges weakly to the equilibrium measure of the spherical model if $\lim_{N \rightarrow \infty} \left(\varphi^{(N)}(\sqrt{N}|z|) \right)^{1/N} = \varphi^{(\infty)}(|z|)$ exist for every point z and coincides with the corresponding characteristic function of the latter model provided $\varphi^{(\infty)}(|z|)$ is continuous at $z = 0$. These statements, which is independent of which order the limits $n \rightarrow \infty$ and $N \rightarrow \infty$ are taken, are proven in Section 4 for $\gamma = d$, $L^d \geq 2$ integer and β different from the critical inverse temperature $\beta_c = \beta_c(d, L)$ of the spherical model.

The following result on the moment generating function holds for admissible coupling matrices, including hierarchical matrix.

Theorem 1.1 *The finite volume moment generating function of the Heisenberg model*

$$\Theta_n^{(N)}(\beta, z) = \int \exp \left(\frac{z}{\sqrt{nN}} \sum_{i=1}^n \sum_{j=1}^N x_{i,j} \right) d\nu_n^{(N)}(\mathbf{x})$$

² $1/N$ plays the role of viscosity since it is in front of the Laplacean as in the hydrodynamic equation of incompressible fluid.

with admissible reflection positive sequence of coupling matrices A , converges

$$\lim_{n, N \rightarrow \infty} \Theta_n^{(N)}(\beta, z) = \Theta(\beta, z) \quad (1.8)$$

to the spherical model moment generating function $\Theta(\beta, z)$ (see (2.34)) as n, N goes to infinity in any order, for $\beta < \beta_c(A)$ given by (2.21) and uniformly in compact intervals of $z \in \mathbb{R}$.

2 Spherical Model

We review the solution and some basic properties of Berlin–Kac model with a positive definite coupling matrix A satisfying a condition stated in (2.12). Formulas written in this section are independent on whether translational invariance holds and are, in addition, suitable to the hierarchical coupling matrix investigated in the next section. We shall make most of those expressions explicit by choosing A the usual discrete Laplacean, denoted here by $-\Delta$. The same symbol will be used for the hierarchical Laplacean in Section 3.

2.1 The Free Energy

Given $\beta \geq 0$ and a positive coupling matrix $J = [J_{ij}]_{i,j=1}^n$, the spherical model of Berlin and Kac [BK] associated with β and J is defined by the partition function

$$Q_n(\beta, J) = \frac{1}{S_n} \int d\sigma_n(\mathbf{x}; \sqrt{n}) \exp \left\{ \frac{-\beta}{2} (\mathbf{x}, J\mathbf{x}) \right\} \quad (2.1)$$

where $(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^n x_i y_i$ denotes the inner product in \mathbb{R}^n ,

$$d\sigma_n(\mathbf{x}; r) = \delta(\|\mathbf{x}\| - r) \prod_{i=1}^n dx_i \quad (2.2)$$

the uniform measure on the sphere $\Sigma_n(r) = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|^2 = (\mathbf{x}, \mathbf{x}) = r^2\}$ of radius r and

$$S_n = \int d\sigma_n(\mathbf{x}; \sqrt{n}) = \frac{2\pi^{n/2} n^{(n-1)/2}}{\Gamma(n/2)} \quad (2.3)$$

is the surface area of the sphere $\Sigma_n(\sqrt{n})$.

The most common choice of coupling matrix J is given by the discrete Laplacean, $-\Delta_\Lambda$,³ on a d -dimensional hypercube $\Lambda \subset \mathbb{Z}^d$ of size $n = L^d$ with periodic boundary condition:

$$-(\Delta_\Lambda f)_i = \sum_{j: |i-j|=1} (f_i - f_j) = 2df_i - \sum_{j: |i-j|=1} f_j \quad (2.4)$$

with the summation over the lattice sites j which are at unit Euclidean distance from $i \in \Lambda$.

³For simplicity, we drop the subindex of Laplacean Δ_Λ if no confusion exists.

Remark 2.1 *The Berlin–Kac model incorporates essential features of the ferromagnetic Ising model, exhibits a phase transition and has the advantage to be exactly solvable in any dimension. See [BK] for an extensive discussion on the thermodynamic properties above and below the critical temperature. The phase transition on the spherical model is of the same nature of that observed in the free Bose gas in which condensation of a single mode occurs (see e.g. [P]). Disordered mean spherical model and equivalence of ensembles has been investigated by Pastur (see [KKPS] and references therein). See also Perez–Wreszinski–van Hemmen [PWH] for disordered spherical models.*

To solve the spherical model it is convenient to introduce an auxiliary expression

$$\begin{aligned} I_n &= \frac{1}{S_n} \int_{\mathbb{R}^n} \prod_{i=1}^n dx_i \exp \left\{ \frac{-1}{2} (\mathbf{x}, (J - \mu) \mathbf{x}) \right\} \\ &= \frac{1}{S_n} \int_0^\infty dr \int d\sigma_n(\mathbf{x}; r) \exp \left\{ \frac{-1}{2} (\mathbf{x}, (J - \mu) \mathbf{x}) \right\} \end{aligned} \quad (2.5)$$

which, after changing variables $r = \sqrt{n}s$ and $\mathbf{x} = s\mathbf{y}$, can be written as

$$I_n = \sqrt{n} \int_0^\infty \frac{ds}{s} \exp \{n h_n(s)\} \quad (2.6)$$

where

$$h_n(s) = \frac{\mu}{2} s^2 + \ln s + f_n(s^2) \quad (2.7)$$

and

$$f_n(s^2) = \frac{1}{n} \ln Q_n(s^2, J) \quad (2.8)$$

is the finite volume free energy of the spherical model.

I_n may be think as the grand–canonical partition function with the Lagrange multiplier $\mu < 0$ playing the role of a chemical potential. The function I_n can be integrated

$$I_n = \frac{2^{n/2-1} \Gamma(n/2)}{n^{(n-1)/2} \sqrt{\det(J - \mu)}} \quad (2.9)$$

and equations (2.6) and (2.9) used to evaluate the free energy when $n \rightarrow \infty$.

For instance, if J is given by (2.4), Fourier spectral analysis (see [D]) can be used in (2.9) together with Stirling's formula $\Gamma(n/2) \sim (n/2e)^{n/2}$ to write

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln I_n = -\frac{1}{2} - \frac{1}{2} \mathbb{E} \ln(-\Delta - \mu) \quad (2.10)$$

where

$$\begin{aligned} \mathbb{E} \ln(-\Delta - \mu) &= \lim_{L \rightarrow \infty} \frac{1}{L^d} \sum_{\substack{m \in \mathbb{Z}^d: \\ -L/2 < m_l \leq L/2}} \ln(\omega(2\pi m/L) - \mu) \\ &= \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} d^d k \ln(\omega(k) - \mu) \end{aligned} \quad (2.11)$$

and

$$\omega(k) = 4 \sum_{l=1}^d \sin^2 \frac{k_l}{2}.$$

Note that, as $n \rightarrow \infty$, $-\Delta$ is unitarily equivalent to an operator of multiplication by $\omega(k)$ in the space $L_2 \left([-\pi, \pi]^d, \mathbb{C} \right)$ of square integrable functions $f : [-\pi, \pi]^d \rightarrow \mathbb{C}$.

Let us now state our assumption on J and explain the probabilistic notation $\mathbb{E}(\cdot)$ in (2.10).

Definition 2.2 *A sequence $A = \{A_n\}_{n \geq 1}$ of coupling matrices (n indicates the order of A_n) is an admissible sequence if each A_n is nonnegative ($(A_n)_{ij} \geq 0$) positive definite real symmetric matrix and*

$$\mathbb{E}f(A) \equiv \lim_{n \rightarrow \infty} \frac{1}{n} \text{Tr}f(A_n) \quad (2.12)$$

exists for every continuous bounded function f .

We require, in addition, that $\mathbf{1} = (1/\sqrt{n})(1, \dots, 1)$ is an eigenvector of A_n with associate eigenvalue 0.

From now on, only admissible sequences of coupling matrices will be considered. By definition,

$$\mathbb{E}f(A) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n f\left(\lambda_i^{(n)}\right) = \lim_{n \rightarrow \infty} \int d\rho_n(\lambda) f(\lambda) = \int d\rho(\lambda) f(\lambda) \quad (2.13)$$

is the expectation with respect to the empirical distribution ρ which is the weak limit of the integrated density of eigenvalues $\lambda_1^{(n)}, \dots, \lambda_n^{(n)}$ (counting multiplicity) of A_n :

$$\rho_n(\lambda) = \frac{1}{n} \sum_{i=1}^n \chi_{[\lambda_i^{(n)}, \infty)}(\lambda) \quad (2.14)$$

with $\chi_{[a,b)}(\lambda) = 1$ if $\lambda \in [a, b)$ and $\chi_{[a,b)}(\lambda) = 0$ otherwise.

The empirical measure ρ associated with $-\Delta$ is absolutely continuous with respect to the Lebesgue measure $d\rho(\lambda) = \rho'(\lambda)d\lambda$,

$$\rho'(\lambda) = \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \Im \left((-\Delta - \lambda - i\varepsilon)_{00}^{-1} \right)$$

exists for almost every λ and is supported in the interval $[0, 4d]$. For $d = 1$, we have explicitly $\rho'(\lambda) = 1 / (\pi \sqrt{4\lambda - \lambda^2})$.

On the other hand, Laplace's asymptotic method applied to (2.6) gives

$$I_n = \sqrt{\frac{2\pi}{-h''(\bar{s})}} \frac{1}{\bar{s}} \exp \{n h(\bar{s})\} (1 + O(1/n)) \quad (2.15)$$

with \bar{s} the value at which $h(s) = \lim_{n \rightarrow \infty} h_n(s)$ attains its maximum value. The existence of a unique strictly positive maximum \bar{s} follows from certain facts that are independent of J considered. Writing $Q_n(s^2) = Q_n(s^2, J)$, we have

1. $Q_n(0) = 1$

2. $Q'_n(s^2) = \frac{-1}{2S_n} \int d\sigma_n(\mathbf{x}; \sqrt{n}) (\mathbf{x}, J\mathbf{x}) \exp \left\{ \frac{-s^2}{2} (\mathbf{x}, J\mathbf{x}) \right\}$

- 3.

$$0 \leq Q''_n(s^2)Q_n(s^2) - (Q'_n(s^2))^2 = \frac{1}{8S_n^2} \int d\sigma_n(\mathbf{x}; \sqrt{n}) d\sigma_n(\mathbf{y}; \sqrt{n}) \\ \times ((\mathbf{x}, J\mathbf{x}) - (\mathbf{y}, J\mathbf{y}))^2 \exp \left\{ \frac{-s^2}{2} [(\mathbf{x}, J\mathbf{x}) + (\mathbf{y}, J\mathbf{y})] \right\}$$

If J satisfies (2.12) then $\|J\| \leq c$ is bounded (when $J = -\Delta$, for instance, we have

$$0 \leq \frac{1}{2} (\mathbf{x}, J\mathbf{x}) \leq \frac{1}{2} \|J\| \|\mathbf{x}\|^2 = 2dn ,$$

in view of (2.2) and $\|J\| = \sup_{k \in [-\pi, \pi]^d} \omega(k) = 4d$ and together with the mean value theorem, we conclude $f_n(s^2) = f'_n(\tilde{s}^2) s^2$ holds for some $0 < \tilde{s} < s$, with

$$-\frac{c}{2} \leq f'_n(s^2) = \frac{1}{n} \frac{Q'_n(s^2)}{Q_n(s^2)} \leq 0$$

and

$$f''_n(s^2) = \frac{1}{n} \left(\frac{Q''_n(s^2)}{Q_n(s^2)} - \left(\frac{Q'_n(s^2)}{Q_n(s^2)} \right)^2 \right) \geq 0$$

uniformly in n . $\{f_n(s^2)\}$ is a sequence of convex functions, uniformly bounded in every compact of \mathbb{R}_+ and there is a subsequence $\{f_{n_j}(s^2)\}$ such that $\lim_{j \rightarrow \infty} f_{n_j}(s^2) = f(s^2)$ exists with $f(s^2)$ convex and differentiable at almost every $s^2 \in \mathbb{R}_+$ (See [KT] and Section 3.1 of [KKPS]). As a consequence, the maximum of h is attained at

$$\bar{s}^2 = \bar{s}^2(\mu) = \frac{-1}{\mu + 2f'(\bar{s}^2)} \quad (2.16)$$

for every $\mu \leq 0$.

Equating (2.10) and (2.15) together with (2.7) yields

$$\frac{\mu}{2} \bar{s}^2 + \ln \bar{s} + f(\bar{s}^2) = -\frac{1}{2} - \frac{1}{2} \mathbb{E} \ln (J - \mu)$$

which, by the inverse function theorem, can be differentiated with respect to μ :

$$\bar{s}^2(\mu) = \mathbb{E} (J - \mu)^{-1} \quad (2.17)$$

in view of $h'(\bar{s}) = 0$. When $J = -\Delta$, the equation reads

$$\bar{s}^2(\mu) = \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} d^d k \frac{1}{\omega(k) - \mu} . \quad (2.18)$$

Solving (2.17) for $\mu = \mu(\bar{s}^2)$, with \bar{s}^2 replaced by an arbitrary positive number β , substituting back to the previous equation gives the free energy of the spherical model

$$f(\beta) = -\mu(\beta)\frac{\beta}{2} - \ln(\sqrt{e\beta}) - \frac{1}{2}\mathbb{E}\ln(J - \mu(\beta)) \quad (2.19)$$

(for $J = -\Delta$ the last term is given by (2.11)).

Remark 2.3 *The sum-rule (2.17) can be obtained directly from the grand-canonical partition function (2.5) (see e.g. [P])*

$$s^2 = \lim_{n \rightarrow \infty} \frac{2}{n} \frac{\partial \ln I_n}{\partial \mu}$$

and this expresses the equivalence between different ensembles of this model. As a function of $\mu \in (-\infty, 0]$, $H_n(\mu) = (2/n) \partial \ln I_n / \partial \mu$ is convex, monotone increasing with $H_n(-\infty) = 0$ and $\lim_{\mu \uparrow 0} H_n(\mu) = \infty$. If \mathcal{P}_0 projects on the invariant subspace of $-\Delta$ associated with $\lambda = 0$, it follows

$$\begin{aligned} s_0^2 &\equiv \mathbb{E} \mathcal{P}_0 (-\Delta - \mu I)^{-1} \\ &= s^2 - \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d \setminus \{0\}} d^d k \frac{1}{\omega(k) - \mu} \\ &\geq s^2 - \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} d^d k \frac{1}{\omega(k)} = s^2 - \mathbb{E}(-\Delta)^{-1} \end{aligned} \quad (2.20)$$

The 0 eigenvalue is said to condensate if $s_0^2 > 0$. According to the above inequality, $s_0^2 > 0$ provided $\beta = s^2 > \mathbb{E}(-\Delta)^{-1}$. Since (2.17) has a unique solution $\mu = \mu(\bar{s})$ with $s_0^2 = 0$ for $s^2 \leq \mathbb{E}(-\Delta)^{-1}$, the spherical model exhibit a phase transition of Bose-Einstein type whenever $\mathbb{E}(-\Delta)^{-1}$ is finite, i. e. when $d > 2$. Note that, for k near 0, $\omega(k) \sim |k|^2$ and

$$\int_{\varepsilon}^1 \frac{1}{k^2} k^{d-1} dk = \begin{cases} (1 - \varepsilon^{d-2}) / (d-2) & \text{if } d \neq 2 \\ \ln 1/\varepsilon & \text{if } d = 2 \end{cases}$$

has no limit for $d \leq 2$. For any sequence A of admissible coupling matrices, the critical inverse temperature of the spherical model is defined by

$$\beta_c(A) = \mathbb{E} A^{-1} \quad (2.21)$$

2.2 Moments Generating Function

We state the main result of this section.

Theorem 2.4 *The block spin random variable*

$$X_n = \frac{1}{\sqrt{n}} \sum_{i=1}^n x_i \quad (2.22)$$

of a spherical model with admissible sequence $J = (J_n)_{n \geq 1}$ of coupling matrices, converge in distribution to a Gaussian variable of zero mean and variance $-1/\mu$ where $\mu = \mu(\beta)$ is the solution of (2.17) with $\bar{s}^2 = \beta$.

Remark 2.5 Since μ approaches 0 from below as $\beta \uparrow \beta_c$, the variance $-1/\mu$ diverges and Theorem 2.4 does not hold anymore. Theorem 2.4 holds for cases (e.g. for no translational invariant J 's) in which the general result of Newman [N] on Gibbs measure satisfying FKG property with finite susceptibility χ cannot be applied.

Let

$$Q_n(\beta, z, \mathbf{h}, J) = \frac{1}{S_n} \int d\sigma_n(\mathbf{x}; \sqrt{n}) \exp \left\{ \frac{-\beta}{2} (\mathbf{x}, J\mathbf{x}) + z (\mathbf{h}, \mathbf{x}) \right\} \quad (2.23)$$

be the partition function of the spherical model including a magnetic field $\mathbf{h} = (h_1, \dots, h_n)$. Since $Q_n(\beta, z, \mathbf{h}, J) = Q_n(\beta, -z, \mathbf{h}, J)$, (2.23) is an even function of z and $Q_n(\beta, 0, \mathbf{h}, J) = Q_n(\beta, J)$ is given by (2.1). One easily verifies that the ratio of partition functions

$$\Theta_n(\beta, z) = \frac{Q_n(\beta, z, \mathbf{1}, J)}{Q_n(\beta, J)} \quad (2.24)$$

with \mathbf{h} in (2.23) given by the n -component unit vector $\mathbf{1} = (1/\sqrt{n})(1, \dots, 1)$, generates the moments of X_n :

$$\frac{\partial^p \Theta_n}{\partial z^p}(\beta, 0) = \frac{\int d\sigma_n(\mathbf{x}; \sqrt{n}) X_n^p \exp \left\{ -\frac{\beta}{2} (\mathbf{x}, J\mathbf{x}) \right\}}{\int d\sigma_n(\mathbf{x}; \sqrt{n}) \exp \left\{ -\frac{\beta}{2} (\mathbf{x}, J\mathbf{x}) \right\}} \equiv \langle X_n^p \rangle$$

and the same procedure of the previous subsection can be used to evaluate the moment generating function Θ_n . Despite of fact that \mathbf{h} in this case goes to $\mathbf{0}$ when $n \rightarrow \infty$, the ratio (2.24) converges to a nontrivial ($\neq 1$) function of z as we shall see in the following.

Repeating (2.5) - (2.9) with I_n replaced by an auxiliary function for (2.23) yields

$$\begin{aligned} K_n &= \frac{1}{S_n} \int_{\mathbb{R}^n} \prod_{i=1}^n dx_i \exp \left\{ \frac{-1}{2} (\mathbf{x}, (J - \kappa) \mathbf{x}) + (\mathbf{h}, \mathbf{x}) \right\} \\ &= \frac{2^{n/2-1} \Gamma(n/2)}{n^{(n-1)/2} \sqrt{\det(J - \kappa)}} \exp \left\{ \frac{1}{2} \left(\mathbf{h}, \frac{1}{J - \kappa I} \mathbf{h} \right) \right\} \\ &= \sqrt{n} \int_0^\infty \frac{ds}{s} \exp \{ n g_n(s) \} \end{aligned} \quad (2.25)$$

with

$$g_n(s) = \frac{\kappa}{2} s^2 + \ln s + f_n(s^2, s)$$

where the free energy

$$f_n(\beta, z) = \frac{1}{n} \ln Q_n(\beta, z; \mathbf{h}, J)$$

is a smooth function of (β, z) such that $f_n(s^2, 0) = f_n(s^2)$ is given by (2.8).

We continue from the last two equations of (2.25):

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{1}{n} \ln K_n &= -\frac{1}{2} - \frac{1}{2} \mathbb{E} \ln (J - \kappa) + \frac{1}{2} \mathbb{E}_{\mathbf{h}} (J - \kappa)^{-1} \\ &= \frac{\kappa}{2} \bar{s}^2 + \ln \bar{s} + f(\bar{s}^2, \bar{s})\end{aligned}\quad (2.26)$$

where $\bar{s} = \bar{s}(\kappa)$ is the solution of equation (2.16) with μ and $f'(\bar{s}^2)$ replaced by κ and the derivative of $f(s^2, s)$ with respect to s^2 (recall $f(s^2, s)$ is an even function of s):

$$\bar{s}^2 = - \left(\kappa + 2 \frac{d}{ds^2} f(s^2, s) \Big|_{s=\bar{s}} \right)^{-1} \quad (2.27)$$

and, if $P_{\mathbf{h}} = \mathbf{h} \mathbf{h}^T / \|\mathbf{h}\|^2$ denotes the projector in the \mathbf{h} direction and $\{\lambda_j, E_j\}_{j=1}^n$ are the spectral elements of J : $J = \sum_{j=1}^n \lambda_j E_j$, we have

$$\mathbb{E}_{\mathbf{h}} (J - \kappa)^{-1} = \lim_{n \rightarrow \infty} \frac{\|\mathbf{h}\|^2}{n} \text{Tr} \frac{1}{J - \kappa} P_{\mathbf{h}} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \frac{1}{\lambda_j - \kappa} \|E_j \mathbf{h}\|^2. \quad (2.28)$$

When $J = -\Delta$, Fourier analysis gives

$$\mathbb{E}_{\mathbf{h}} (-\Delta - \kappa)^{-1} = \frac{\beta}{(2\pi)^d} \int_{[-\pi, \pi]^d} d^d k \frac{|\hat{h}(\mathbf{k})|^2}{\omega(\mathbf{k}) - \kappa}$$

with

$$\hat{h}(\mathbf{k}) = \sum_{x \in \mathbb{Z}^d} h_x \exp(i\mathbf{k} \cdot \mathbf{x}).$$

Since $\mathbf{1} = (1/\sqrt{n})(1, \dots, 1)$ is the unique eigenvector of (2.4) associated with the eigenvalue 0, only the zero-mode contributes to the expectation (2.28) with $\mathbf{h} = \mathbf{1}$:

$$\mathbb{E}_{\mathbf{h}} (-\Delta - \kappa)^{-1} = (\mathbf{h}, (-\Delta - \kappa)^{-1} \mathbf{h}) = \frac{-z^2}{n\kappa}. \quad (2.29)$$

Assuming $\mathbf{1}$ an eigenvector of J (orthogonal to the complementary space in view of $J = J^T$) with associate eigenvalue $\lambda = 0$ the same result holds with $-\Delta$ replaced by J . We shall continue our calculation of the generating function Θ_n with J satisfying the assumptions of admissible coupling matrices.

The free energy function is obtained by equating the two lines of (2.26) up to order $1/n$ and proceeding as in equations (2.16)–(2.19):

$$n f_n(\beta, z) = -\frac{n}{2} (\beta \kappa + \ln(e\beta) + \mathbb{E} \ln(J - \kappa)) - \frac{z^2}{2\kappa} + O(1), \quad (2.30)$$

where $\kappa_n = \kappa_n(\beta, z)$ is the solution of

$$\beta = \mathbb{E} \frac{1}{-\Delta - \kappa} + \frac{z^2}{n\kappa^2}, \quad (2.31)$$

and the order 1 term is independent of z by (2.15) for K_n together with (2.32) below.

Denoting by $w_n(\kappa, z)$ the r.h.s. of (2.31), w_n is a decreasing function of κ for $\kappa < 0$ with $w_n(-\infty, z) = 0$ and $\lim_{\kappa \rightarrow 0} w_n(\kappa, z) = \infty$. By the implicit function theorem, the solution $\kappa_n(\beta, z)$ of (2.31) is the unique real analytic function of z^2 in a neighborhood of $z^2 = 0$. Consequently, $\kappa_n(\beta, 0) = \mu(\beta)$ for every n , $\lim_{n \rightarrow \infty} \kappa_n(\beta, z) = \mu(\beta)$ uniformly in z , where $\mu(\beta)$ is the solution of (2.17) and

$$\kappa = \mu + \frac{c}{n} z^2 + o\left(\frac{z^2}{n}\right) \quad (2.32)$$

by Taylor theorem, where

$$c = [\mathbb{E}(1 - J/\mu)^{-2}]^{-1}$$

is obtained by plugging the resolvent equation

$$\frac{1}{J - \kappa} - \frac{1}{J - \mu} = (\kappa - \mu) \frac{1}{J - \kappa} \frac{1}{J - \mu}$$

into (2.31) together with (2.17) and (2.32).

Proof of Theorem 2.4. Substituting $\exp(n f_n(\beta, z))$ in the numerator and denominator (with $z = 0$) of (2.24), taking into account that $O(1)$ term in (2.30) does not depend on z , gives

$$\Theta_n(\beta, z) = \exp\left(-n \frac{\beta}{2} (\kappa - \mu) - \frac{n}{2} \mathbb{E} \ln \frac{J - \kappa}{J - \mu} - \frac{z^2}{2\kappa}\right) \left(1 + o\left(\frac{1}{n}\right)\right) \quad (2.33)$$

and it suffices to verify that

$$\ln \Theta(\beta, z) = \lim_{n \rightarrow \infty} \ln \Theta_n(\beta, z) \quad (2.34)$$

converges uniformly in compacts of z and for every $\beta < \beta_c$ to a limit proportional to z^2 . The limit (2.34) exists by the same reasons employed to show existence of $\lim_{n \rightarrow \infty} f_n(s^2)$. Plugging (2.32) into (2.33) together (2.17) gives

$$\ln \Theta(\beta, z) = \frac{-z^2}{2} \left(c\beta - c\mathbb{E} \frac{1}{J - \mu} + \frac{1}{\mu} \right) = \frac{-z^2}{2\mu(\beta)}.$$

□

3 Hierarchical Spherical Model

3.1 Hierarchical Laplacean

Paiva and Perez [PP] have investigated the semi-groups generated by d -dimensional Hierarchical Laplacean $-\Delta$ in the presence of disorder by using, for the first time, spectral analysis. Although $-\Delta$ has discrete spectrum, they have shown that $\exp(t\Delta) \delta_0$, with δ_0 localized at origin, diffuses. We quote [Kr] and references therein for spectral localization in hierarchical Anderson model. Here, in order to apply the limit theorem established in Subsection 2.2 we need the spectral theorem for homogeneous hierarchical matrices. We extend the work of Watanabe [W] to arbitrary spectral dimension d (see Remark 3.4).

Given integer numbers $L, K > 1$ and $d \geq 1$, let

$$\Lambda_K = \{0, 1, \dots, L^K - 1\}^d \subset \mathbb{Z}^d$$

be a hypercube with cardinality $|\Lambda_K| = L^{dK} = n$. Let $\boldsymbol{\theta} = (\theta_1, \dots, \theta_K)$ denote the coordinates of a point $i \in \Lambda_K$ written in the L^d base

$$i = \sum_{k=1}^K \theta_k L^{k-1}, \quad \theta_k \in \{0, 1, \dots, L-1\}^d.$$

From now on we use these coordinates to index components of a vector $u = (u_{\boldsymbol{\theta}})$ in \mathbb{R}^{Λ_K} .

Let $B : \mathbb{R}^{\Lambda_K} \longrightarrow \mathbb{R}^{\Lambda_{K-1}}$ be the block operator

$$(Bu)_{\boldsymbol{\tau}} = \frac{1}{L^{d/2}} \sum_{\boldsymbol{\theta} \in \{0, 1, \dots, L^K - 1\}^d} u_{(\boldsymbol{\theta}, \boldsymbol{\tau})} \quad (3.1)$$

and let $B^* : \mathbb{R}^{\Lambda_{K-1}} \longrightarrow \mathbb{R}^{\Lambda_K}$ be its adjoint

$$(B^*v, u)_{\Lambda_K} = (v, Bu)_{\Lambda_{K-1}}$$

with respect to the inner product $(u, w)_{\Lambda_K} = \sum_{\boldsymbol{\theta}} u_{\boldsymbol{\theta}} w_{\boldsymbol{\theta}}$:

$$(B^*v)_{(\boldsymbol{\theta}, \boldsymbol{\tau})} = \frac{1}{L^{d/2}} v_{\boldsymbol{\tau}}.$$

Define a real symmetric matrix in \mathbb{R}^n

$$J = \sum_{k=1}^K L^{-2k} (B^*)^k B^k. \quad (3.2)$$

Proposition 3.1 *The associate quadratic form of J for $L^d = 2$ and $d = 2/(\alpha - 1)$ gives the hierarchical energy*

$$\begin{aligned} -H &= \sum_{k=1}^K 2^{-\alpha k} \sum_{r=1}^{2^{K-k}} (S_{k,r})^2 \\ S_{k,r} &= \sum_{(r-1)2^k < j \leq r2^k} \sigma_j \end{aligned}$$

introduced by Dyson [D] in his study of the Ising model with $1/|i - j|^\alpha$ -interaction.

Remark 3.2 *Note that $\alpha(d) = (d + 2)/d$ ranges from 2 to 1 as d varies from 2 to ∞ .*

Proof. Setting $L^d = 2$, we have $S_{k,r} = 2^{k/2} (B^k \sigma)_\theta$ for some $\theta \in \Lambda_{K-k}$ and

$$-H = \sum_{k=1}^K 2^{-\alpha k + k} (B^k \sigma, B^k \sigma)_{\Lambda_{K-k}} = \sum_{k=1}^K L^{-2k} \left(\sigma, (B^*)^k B^k \sigma \right)_{\Lambda_K} = (\sigma, J\sigma)_{\Lambda_K}$$

as claimed. □

We require the hierarchical Laplacean $-\Delta$ satisfies

$$-\Delta \mathbf{1} = 0 \tag{3.3}$$

(Theorem 3.3 below shows that 0 is also a simple eigenvalue). Writing

$$J = \Delta + \mu_0 I$$

together with (3.3), (3.2) and (3.1), we have

$$\mu_0 = (\mathbf{1}, J\mathbf{1})_{\Lambda_K} = \frac{1}{L^{dK}} \sum_{k=1}^K L^{-2k} (B^k \mathbf{1}, B^k \mathbf{1})_{\Lambda_{K-k}} = \sum_{k=1}^K L^{-2k} \tag{3.4}$$

and

$$-\Delta = \sum_{k=1}^K L^{-2k} \left(- (B^*)^k B^k + I \right) \tag{3.5}$$

generates a stochastic semi-group.

Now we observe that

$$B^k (B^*)^k = BB^* = I \tag{3.6}$$

holds for every $k = 1, \dots, K$ and

$$P_k = (B^*)^k B^k \tag{3.7}$$

is an orthogonal $P_k = P_k^*$ projection matrix $P_k^2 = P_k$ on the subspace of vectors in \mathbb{R}^{Λ_K} which assumes constant value over blocks of size L^{dk} . It follows from (3.6)

$$\begin{aligned} P_j P_k &= (B^*)^j B^j (B^*)^k B^k = (B^*)^j B^{j-k} B^k = P_j \\ P_k P_j &= (B^*)^k B^k (B^*)^j B^j = (B^*)^k (B^*)^{j-k} B^j = P_j \end{aligned} \tag{3.8}$$

hold for any $j > k$ and we have the following inclusions

$$P_K < P_{K-1} < \dots < P_1 < P_0 \equiv I \tag{3.9}$$

in the sense that $A < B$ if, and only if, $(u, Au)_{\Lambda_k} < (u, Bu)_{\Lambda_k}$ holds for all $u \in \mathbb{R}^{\Lambda_K}$.

Let

$$Q_k = P_k - P_{k+1} \tag{3.10}$$

for $k = 0, 1, \dots, K-1$ and

$$Q_K = P_K$$

be the block fluctuation operator.

Theorem 3.3 (Spectral) *The collection $\{Q_k\}_{k=0}^K$ of $n \times n$ real orthogonal projection matrices*

$$Q_j Q_k = \delta_{jk} Q_k \quad (3.11)$$

are the spectral partition of unit

$$I = \sum_{k=0}^K Q_k$$

and

$$f(-\Delta) = \sum_{k=0}^K f(\lambda_k) Q_k \quad (3.12)$$

holds with

$$\lambda_k = \frac{L^{-2k} - L^{-2K}}{L^2 - 1} \quad (3.13)$$

for any continuous function $f : [0, 1/(L^2 - 1)] \rightarrow \mathbb{R}$. It follows that $-\Delta$ is a positive definite matrix where λ_k , $k = 0, \dots, K-1$, is an eigenvalue of multiplicity $L^{d(K-k)}(1 - L^{-d})$ and $\lambda_K = 0$ a simple eigenvalue.

Proof. The prove is essentially given in [W]. By (3.10) and (3.8)

$$\begin{aligned} Q_j Q_k &= (P_j - P_{j+1})(P_k - P_{k+1}) \\ &= P_j(P_k - P_{k+1}) - P_{j+1}(P_k - P_{k+1}) \\ &= (P_j - P_{j+1}) - (P_{j+1} - P_{j+2}) = 0 \end{aligned}$$

for any $k < j < K$ and the same holds for $j < k < K$. For $j < k = K$,

$$Q_j Q_K = (P_j - P_{j+1}) P_K = P_K - P_K = 0$$

and for $j = k$

$$Q_k Q_k = (P_k - P_{k+1})(P_k - P_{k+1}) = P_k + P_{k+1} - 2P_{k+1} = Q_k.$$

By definition,

$$\sum_{k=0}^K Q_k = \sum_{k=0}^{K-1} (P_k - P_{k+1}) + P_K = P_0 - P_K + P_K = I.$$

Finally, by (3.5), (3.7) and (3.10), we have

$$\begin{aligned} -\Delta &= \sum_{j=1}^K L^{-2j} (-P_j + I) \\ &= \sum_{j=1}^K L^{-2j} \sum_{k=0}^{j-1} Q_k \\ &= \sum_{k=0}^{K-1} \left(\sum_{j=k+1}^K L^{-2j} \right) Q_k + 0 \cdot Q_K \end{aligned} \quad (3.14)$$

which gives (3.12) with $f(x) = x$. It follows by (3.11) that (3.12) holds for any polynomial and, by Weierstrass approximation theorem, for any uniformly continuous function.

Since P_k projects on vectors in Λ_K which are constant over disjoint blocks of size L^{dk} , the rank of P_k is

$$\text{rank} P_k = L^{d(K-k)} .$$

By definition (3.10) together with the inclusions (3.9), the rank of the block fluctuation projector Q_k is

$$\text{rank} Q_k = L^{d(K-k)} - L^{d(K-k-1)} \quad (3.15)$$

for $k = 1, \dots, K-1$ and

$$\text{rank} Q_K = 1$$

and these concludes the prove of Theorem 3.3. □

Remark 3.4 *The spectral measure μ^K associated with the vector $\delta_{\theta} = (\delta_{\theta,j})_{j \in \Lambda_K}$ defined by*

$$(\delta_{\theta}, f(-\Delta) \delta_{\theta}) = \int_{-\infty}^{\infty} f(x) d\mu_{\theta}^K(x)$$

for every bounded Borel function $f : \mathbb{R} \rightarrow \mathbb{C}$, is given by

$$d\mu^K(x) = \sum_{k=0}^{K-1} \frac{L^d - 1}{L^{d(k+1)}} \delta(x - \lambda_k) dx + \frac{1}{L^{dK}} \delta(x) dx ,$$

by inspection of the matrix elements $Q_{\theta\theta'}$, and is independent of θ . As $n = L^{dK}$ tends to infinity, μ^{∞} is the unique weak- $$ limit point of the corresponding empirical measure (2.14) (see Theorem 1.2 of Kritchevski [Kr]). The number*

$$d := 2 \lim_{t \downarrow 0} \frac{\ln \mu^{\infty}([0, t])}{\ln t}$$

is called spectral dimension of $-\Delta$.

3.2 The Free Energy

To compute the free energy (2.19) of the spherical model associated with $\beta > 0$ and hierarchical Laplacean matrix $J = -\Delta$ we need to evaluate the expectation \mathbb{E} with respect to the empirical measure of eigenvalues of $-\Delta$ in both the last term of the r.h.s. of (2.19) and in the implicit equation (2.17) for $\mu = \mu(\beta)$.

Using Theorem 3.3 together with (2.13) and linearity of trace, we have

$$\begin{aligned} \frac{1}{L^{dK}} \text{Tr} f(-\Delta) &= \frac{1}{L^{dK}} \sum_{k=0}^K f(\lambda_k) \text{Tr} Q_k \\ &= (1 - L^{-d}) \sum_{k=0}^{K-1} L^{-dk} f(\lambda_k) + \frac{1}{L^{dK}} f(\lambda_K) \end{aligned}$$

in view of the fact that the eigenvalues of Q_k are 0 and 1 together with (3.15). Hence, the subsequence $\{f_{n_K}(s^2)\}_{K \in \mathbb{N}}$ of the free energy with $n_K = L^{dK}$ converges to (2.19) where

$$\mathbb{E} \ln(-\Delta - \mu) = (1 - L^{-d}) \sum_{k=0}^{\infty} L^{-dk} \ln \left(\frac{L^{-2k}}{L^2 - 1} - \mu \right) \quad (3.16)$$

and $\mu = \mu(\beta)$ solves

$$\begin{aligned} \beta &= \mathbb{E}(-\Delta - \mu I)^{-1} \\ &= (1 - L^{-d}) \sum_{k=0}^{\infty} L^{-dk} \frac{L^2 - 1}{L^{-2k} - \mu(L^2 - 1)} + \rho_0 \end{aligned} \quad (3.17)$$

including the 0-eigenvalue contribution $\rho_0 = \mathbb{E} \mathcal{P}_0(-\Delta - \mu I)^{-1}$ which, by Remark 2.3, may have macroscopic occupation.

Analogously to (2.20), we have

$$\begin{aligned} \rho_0 &\geq \beta - (1 - L^{-d}) (L^2 - 1) \sum_{k=0}^{\infty} L^{-(d-2)k} \\ &= \beta - \frac{(1 - L^{-d}) (L^2 - 1)}{1 - L^{-d+2}} \end{aligned}$$

which is strictly positive provided $d > 2$ and $\beta > \beta_c(d, L)$ where

$$\beta_c(d, L) = \frac{(1 - L^{-d}) (L^2 - 1)}{1 - L^{-d+2}}$$

is the critical inverse temperature of the hierarchical spherical model.

Remark 3.5 *The geometric multiplicity $(L^d - 1)L^{d(K-k-1)}$ of each eigenvalue λ_k , $k = 0, \dots, K-1$, of the hierarchical Laplacean (3.14) can be lifted by fluctuation projectors $Q_{k,\theta}$ depending on the index $\theta \in \Lambda_{K-k}$, such that*

$$Q_{k,\theta} Q_{k,\theta'} = \delta_{\theta\theta'} Q_{k,\theta} ,$$

and a nonhomogeneous Laplacean can be defined by

$$-\Delta^{\text{nh}} = \sum_{k=0}^K \sum_{\theta \in \Lambda_{K-k}} \lambda_{k,\theta} Q_{k,\theta}$$

with

$$\lambda_{k,\theta} = c \lambda_k \exp \{X_{k,\theta}\}$$

where λ_k is given by (3.13), $\{X_{k,\theta}\}$ chosen according to a common probability distribution \mathbb{P} with mean 0 with $c^{-1} = \mathbb{E} \exp(X_{k,\theta}) < \infty$. In this case, we have

$$\begin{aligned} \frac{1}{L^{dK}} \text{Tr} f(-\Delta) &= (1 - L^{-d}) \sum_{k=0}^{K-1} L^{-dk} \frac{1}{L^{d(K-k)}} \sum_{\theta \in \Lambda_{K-k}} f(c \lambda_k \exp \{X_{k,\theta}\}) + \frac{1}{L^{dK}} f(\lambda_K) \\ &\longrightarrow (1 - L^{-d}) \sum_{k=0}^{\infty} L^{-dk} \mathbb{E} f(c \lambda_k \exp \{X_{k,0}\}) = \mathbb{E} f(-\Delta) \end{aligned}$$

for almost every $\{X_{k,0}\}$ with respect to distribution \mathbb{P} , by the law of large numbers. Here $\mathbb{E}f$ denotes the expectation with respect to the product measure $d\rho(\lambda)\mathbb{P}(dx)$ with ρ the empirical distribution relative to the eigenvalues $\{\lambda_k\}$ and \mathbb{P} the common distribution of variables $X_{k,\theta}$. Expressions (2.19), (3.16) and (3.17) are obtained accordingly. For instance,

$$\mathbb{E}(-\Delta - \mu I)^{-1} = (1 - L^{-d}) \sum_{k=0}^{\infty} L^{-dk} \mathbb{E} \frac{L^2 - 1}{cL^{-2k} \exp\{X_{k,0}\} - \mu(L^2 - 1)} + \rho_0.$$

3.3 Continuum Hierarchical Laplacean

Hierarchical Laplaceans have discrete eigenvalues. We shall now consider a continuous version obtained by a limit procedure.

We shall take $L \downarrow 1$ simultaneously to $K \rightarrow \infty$ maintaining $K \ln L$ fixed equal to $C \in \mathbb{R}_+ \cup \{\infty\}$. Equation (3.17), for instance, reads

$$\begin{aligned} \mathbb{E}(-\Delta - \mu I)^{-1} &= \lim_{L \downarrow 1} (1 - L^{-d}) \sum_{k=0}^{K(L)} L^{-dk} \left(\frac{L - 1}{(L^2 - 1)} (L^{-2k} - L^{-2K}) - \mu \right)^{-1} \\ &= d \int_0^C \frac{2}{\exp(-2y) - \exp(-2C) - 2\mu} e^{-dy} dy \\ &= \int_0^{(1 - \exp(-2C))/2} \frac{1}{\lambda - \mu} d\rho(\lambda) \end{aligned} \quad (3.18)$$

where $d\rho(\lambda) = \rho'(\lambda)d\lambda$ is absolutely continuous with respect to the Lebesgue measure $d\lambda$ with

$$\rho'(\lambda) = 2^{d/2} \frac{d}{2} \left(\lambda + \frac{\exp(-2C)}{2} \right)^{d/2-1}$$

if $\lambda \in [0, (1 - \exp(-2C))/2]$ and 0 otherwise. So, the empirical measure for the eigenvalues of the hierarchical Laplacean $-\Delta$ with $L \downarrow 1$ converges provided $K = K(L)$ increases faster than $C(\ln L)^{-1}$. We take $C = \infty$, for simplicity.

Accordingly, equations (2.19), (3.16) and (3.17) holds with λ_k replaced by $\lambda_k(L - 1)$ and

$$\begin{aligned} \rho_0 &\geq \beta - \lim_{L \downarrow 1} \frac{(1 - L^{-d})(L^2 - 1)}{(1 - L^{-d+2})(L - 1)} \\ &= \beta - \frac{2d}{d - 2} \end{aligned}$$

has a strictly positive limit provided $d > 2$ and $\beta > \beta_c(d)$ where

$$\beta_c(d) = \frac{2d}{d - 2}. \quad (3.19)$$

Note that

$$\mathbb{E}(-\Delta)^{-1} = \int_0^{1/2} \lambda^{-1} d\rho(\lambda) = 2^{d/2} \frac{d}{2} \int_0^{1/2} \lambda^{d/2-2} d\lambda = \frac{2d}{d - 2}$$

and a phase transition of Bose–Einstein condensation type occur at the critical inverse temperature $\beta_c = \mathbb{E}(-\Delta)^{-1}$ as in the spherical model with the usual Laplacean interaction.

We now compute integral (3.18) with $C = \infty$ and $d = 4$, for comparison purposes. For $\mu \leq 0$, we continue

$$\mathbb{E}(-\Delta - \mu I)^{-1} = 8 \int_0^{1/2} \frac{\lambda}{\lambda - \mu} d\lambda = 4 \left(2\mu \ln \left(1 - \frac{1}{2\mu} \right) + 1 \right)$$

and equation (2.17) with Δ given by the hierarchical Laplacean at $d = 4$ and $\bar{s}^2 = \beta$, reads

$$1 - \frac{\beta}{4} = -2\mu \ln \left(1 - \frac{1}{2\mu} \right). \quad (3.20)$$

4 Convergence to Spherical Model

4.1 The $O(N)$ Heisenberg Model

The partition function of the $O(N)$ symmetric Heisenberg model in a d -dimensional cubic box $\Lambda \subset \mathbb{Z}^d$ of cardinality $n = L^d$ is given by

$$Z_n^{(N)}(\beta, A) = \frac{1}{S_N^n} \int_{\mathbb{R}^{nN}} \exp \left\{ -\frac{\beta}{2} (\mathbf{x}, A\mathbf{x}) \right\} \prod_{j=1}^n d\sigma_0^{(N)}(x_j) \quad (4.1)$$

where $\mathbf{x} = (x_1, \dots, x_n)$ is a n -tuple with each $x_j \in \mathbb{R}^N$, $A = J \otimes I$ is the tensor product of a $n \times n$ coupling matrix J with the $N \times N$ identity matrix I and $\sigma_0^{(N)}(dx) \equiv \sigma_N(dx; \sqrt{N}) / S_N$ is the “a priori” uniform probability measure on the N -dimensional sphere $|x|^2 = N$ of radius \sqrt{N} with surface area S_N given by (2.3). The inner product on $\mathbb{R}^n \otimes \mathbb{R}^N$ is denoted here by $(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^n x_i \cdot y_i$.

The expected value with respect to the Heisenberg measure $\nu_n^{(N)}$ (see (1.1)) is defined by

$$\langle F \rangle_{\nu_n^{(N)}} = \frac{1}{S_N^n Z_n^{(N)}} \int_{\mathbb{R}^{nN}} F(\mathbf{x}) \exp \left\{ -\frac{\beta}{2} (\mathbf{x}, A\mathbf{x}) \right\} \prod_{j=1}^n d\sigma_0^{(N)}(x_j)$$

and from here on, A in $\langle \cdot \rangle_{\nu_n^{(N)}} = \langle \cdot \rangle_{\nu_n^{(N)}}(\beta, A)$ is assumed to be a sequence of admissible coupling matrices in the following sense.

Definition 4.1 *A sequence $A = \{A_n\}_{n \geq 1}$ of coupling matrices is an admissible reflection positive sequence if each A_n is admissible in the sense of Definition 2.2 and $\langle \cdot \rangle_{\nu_n^{(N)}}(\beta, A)$ satisfies the reflection positivity condition (4.4).*

Kunz and Zumbach [KZ] have devised a way of proving convergence of the free energy of the Heisenberg model to the spherical model for nearest neighbor interactions. We shall first show that their method holds for admissible reflection positive coupling matrices and prove Theorem 1.1 afterwards.

Theorem 4.2 *The finite volume free energy of the Heisenberg model*

$$f_n^{(N)}(\beta) = \frac{1}{n \cdot N} \ln Z_n^{(N)}(\beta, A) \quad (4.2)$$

with admissible reflection positive sequence of coupling matrices A , converges

$$\lim_{n, N \rightarrow \infty} f_n^{(N)}(\beta) = f(\beta) \quad (4.3)$$

to the spherical model free energy (2.19) as n, N goes to infinity in any order.

The original proof by Kac and Thompson [KT] asserts that (4.3) holds for all coupling matrix J satisfying translation invariance $J_{ij} = g(i - j)$. It turns out that their proof has a serious gap fixed in [KZ] only for the usual Laplacean given by (2.4). Our presentation, based on an unpublished Appendix of [KZ], uses the Laplace method discussed in Subsection 2.1 and is written for admissible coupling matrices. Consequently, it works for the hierarchical Laplacean matrix coupling (3.2) as well.

Reflection Positivity is the missing ingredient. For a basic exposition of the abstract theory see [FILS]. If P denotes a plane perpendicular to coordinate axes which divides Λ into two halves Λ_{\pm} :

$$\Lambda = \Lambda_+ \cup \Lambda_- \quad \text{and} \quad \Lambda_+ \cap \Lambda_- = \emptyset$$

(P cuts bonds perpendicularly and do not intercept sites of Λ), let

$$r : \Lambda_+ \longrightarrow \Lambda_-$$

be a map which assigns to each $j \in \Lambda_+$ its reflected image $rj \in \Lambda_-$, i.e. the site symmetric with respect to P . The reflection map r induces a linear morphism $\pi_P : \mathfrak{A}_+ \longrightarrow \mathfrak{A}_-$, on the abelian algebra \mathfrak{A}_{\pm} of bounded function on the configuration space $\Omega_{\pm} = \mathbb{R}^{\Lambda_{\pm}}$ given by

$$\pi_P F \left(\{x_j\}_{j \in \Lambda_+} \right) = F \left(\{x_{rj}\}_{j \in \Lambda_+} \right) .$$

By a linear morphism we mean $\pi_P(FG) = \pi_P(F) \pi_P(G)$ is satisfied for any $F, G \in \mathfrak{A}_+$.

Definition 4.3 (Reflection Positivity) *A state $\langle \cdot \rangle_{\nu}$ is said to be a reflection positivity functional if*

$$\langle F \pi_P(F) \rangle_{\nu} \geq 0 \quad (4.4)$$

holds for all $F \in \mathfrak{A}_+$.

According to [FILS], $\langle \cdot \rangle_{\nu_n^{(N)}}$ defined by the expected value with respect to the measure $\nu_n^{(N)}$ given by (1.1) with $A = -\Delta \otimes I$, $-\Delta$ the usual Laplacean with periodic boundary conditions (2.4), is a reflection positivity functional. Once (4.4) holds, we have the Schwarz inequality

$$\langle FG \pi_P(FG) \rangle_{\nu} \leq \langle F \pi_P(F) \rangle_{\nu}^{1/2} \langle G \pi_P(G) \rangle_{\nu}^{1/2} . \quad (4.5)$$

Note that the normalization is unimportant and can be dropped in both sides of the inequality as long as n and N are kept fixed. Applying (4.5) to every plane P which cuts bonds perpendicularly, $\langle \cdot \rangle$ satisfies the chessboard inequality (see e.g. [FILS])

$$\left\langle \prod_j F_j(h_j) \right\rangle_\nu \leq \prod_j \left\langle \prod_i F_i(h_j) \right\rangle_\nu^{1/n} \quad (4.6)$$

where, for each j , $F_j(h, \{x_i\}_{i \in \Lambda}) = F(h, x_j)$ is a one parameter h family of bounded function in \mathbb{R}^N . Note that (4.6) has a homogenization effect.

Proof of Theorem 4.2. Repeating the steps of (2.5) and (2.6), we have

$$\begin{aligned} I_n^{(N)} &= \frac{1}{S_N^n} \int_{\mathbb{R}^{n \cdot N}} \prod_{i=1}^n d^N x_i \exp \left\{ \frac{-1}{2} (\mathbf{x}, (A - \mu) \mathbf{x}) \right\} \\ &= N^{n/2} \int_{\mathbb{R}_+^n} \prod_{j=1}^n ds_j s_j^{N-1} \exp \left\{ \frac{\mu N}{2} \sum_{j=1}^n s_j^2 \right\} Z_n^{(N)}(1, B) \end{aligned} \quad (4.7)$$

where

$$B = SJS \otimes I \quad (4.8)$$

is the coupling matrix $A = J \otimes I$ modified by a matrix $S = \text{diag}(s_1, \dots, s_n)$ with the n -vector $\mathbf{s} = (s_1, \dots, s_n)$ in the diagonal. Applying the chessboard inequality (4.6) to this nonhomogeneous partition function (see [KZ]), yields

$$Z_n^{(N)}(1, B) \leq \prod_{j=1}^n (Z_n^{(N)}(s_j^2, A))^{1/n}$$

and we have an upper bound

$$\begin{aligned} I_n^{(N)} &\leq N^{n/2} \int_{\mathbb{R}_+^n} \prod_{j=1}^n ds_j s_j^{N-1} \exp \left\{ \frac{\mu N}{2} \sum_{j=1}^n s_j^2 \right\} \prod_{j=1}^n (Z_n^{(N)}(s_j^2, A))^{1/n} \\ &= \left(\sqrt{N} \int_0^\infty \frac{ds}{s} \exp \{ N h_n^{(N)}(s) \} \right)^n \end{aligned} \quad (4.9)$$

where, similarly to (2.7),

$$h_n^{(N)}(s) = \frac{\mu}{2} s^2 + \ln s + f_n^{(N)}(s^2) .$$

We are now looking for a lower bound of (4.7). Using the $O(N)$ symmetry together with Jensen inequality, we have

$$\begin{aligned} I_n^{(N)} &\geq N^{n/2} \int_{[c(1-1/N), c]^n} \prod_{j=1}^n ds_j s_j^{N-1} Z_n^{(N)}(1, S(J - \mu I)S \otimes I) \\ &= (NR_N^2)^{n/2} \langle Z_n^{(N)}(1, S(J - \mu I)S \otimes I) \rangle \\ &\geq (NR_N^2)^{n/2} Z_n^{(N)}(1, \langle S(J - \mu I)S \rangle \otimes I) \end{aligned} \quad (4.10)$$

where $\langle \cdot \rangle$ denotes the average with respect to the product measure $\prod_{j=1}^n d\mu_N(s_j)$ with $d\mu_N(s) = ds s^{N-1} \chi_{[c(1-1/N), c]}(s)/R_N$ and $c = a^{1/N} \sqrt{\beta N^{1/N}}$. The constant a is here chosen arbitrarily while it has to be tuned properly for the moment generating function. The normalization

$$\sqrt{N} \beta^{-N/2} R_N = \int_{c(1-1/N)}^c ds s^{N-1} = a \left[1 - \left(1 - \frac{1}{N} \right)^N \right] \longrightarrow a \left(1 - \frac{1}{e} \right)$$

and the two first moments of μ_N ,

$$\langle s \rangle = \int s d\mu_N(s) = \sqrt{\beta} \frac{N^{1+1/2N}}{N+1} \frac{1 - (1 - 1/N)^{N+1}}{1 - (1 - 1/N)^N} \longrightarrow \sqrt{\beta}$$

and

$$\langle s^2 \rangle = \int s^2 d\mu_N(s) = \beta \frac{N^{1+1/N}}{N+2} \frac{1 - (1 - 1/N)^{N+2}}{1 - (1 - 1/N)^N} \longrightarrow \beta$$

are the only quantities that contribute to the lower bound. We have

$$(\mathbf{x}, \langle S(J - \mu I) S \rangle \otimes I \mathbf{x}) = (\mathbf{x}, (\langle s \rangle^2 A - \langle s^2 \rangle \mu) \mathbf{x})$$

which, in view of (4.1), implies

$$\begin{aligned} Z_n^{(N)}(1, \langle S(J - \mu I) S \rangle \otimes I) &= \exp \left(\frac{1}{2} \mu n N \langle s^2 \rangle \right) Z_n^{(N)}(\langle s \rangle^2, A) \\ &= \exp \left\{ n N \left(\frac{1}{2} \mu \langle s^2 \rangle + f_n^{(N)}(\langle s \rangle^2) \right) \right\}. \end{aligned}$$

Similarly to (2.9), we have

$$I_n^{(N)} = \frac{2^{n(N/2-1)} (\Gamma(N/2))^n}{N^{n(N-1)/2} \sqrt{\det(J - \mu I)}}.$$

Taking the limit $\lim_{n, N \rightarrow \infty} \ln I_n^{(N)}/nN$, in any order someone wishes, of both (4.9) and (4.10) yields

$$\frac{\beta}{2} \mu + \frac{1}{2} \ln \beta + f^*(\beta) \leq -\frac{1}{2} - \frac{1}{2} \mathbb{E} \ln(J - \mu) \leq \frac{\mu}{2} \bar{s}^2 + \ln \bar{s} + f^*(\bar{s}^2)$$

where

$$f^*(\beta) = \lim_{n, N \rightarrow \infty} f_n^{(N)}(\beta)$$

and \bar{s} is the solution to equation (2.17).

Now we solve (2.17) with $\bar{s}^2 = \beta$ for $\mu = \mu(\beta)$ and replace this function in the previous equation to obtain a lower bound

$$f^*(\beta) \geq f(\beta)$$

with f given by spherical free energy (2.19).

The other side of the same equation gives the following upper bound

$$f^*(\beta) \leq -\frac{1}{2} \ln(e\beta) + \sup_{\mu < 0} \frac{1}{2} (-\mathbb{E} \ln(J - \mu) - \beta\mu)$$

Note that this inequality holds for any $\mu < 0$, so it holds for μ that gives the least upper bound. The supremum is attained at $\mu = \mu(\beta)$ that solves (2.17) with $\bar{s}^2 = \beta$ and we have the upper bound

$$f^*(\beta) \leq f(\beta)$$

concluding the proof of Kac–Thompson theorem for admissible reflection positive coupling matrices A .

□

The above proof is now modified to establish convergence of the moments generating function.

4.2 Proof of Theorem 1.1

The ratio of partition functions

$$\Theta_n^{(N)}(\beta, z) = \langle \exp(z(\mathbf{1}, \mathbf{x})) \rangle_{\nu_n^{(N)}} = \frac{Z_n^{(N)}(\beta, z, \mathbf{1}, A)}{Z_n^{(N)}(\beta, A)} \quad (4.11)$$

where

$$Z_n^{(N)}(\beta, z, \mathbf{1}, A) = \int \exp \left\{ \frac{-\beta}{2} (\mathbf{x}, A\mathbf{x}) + z(\mathbf{1}, \mathbf{x}) \right\} \prod_{j=1}^n d\sigma_0^{(N)}(x_j) \quad (4.12)$$

with $\mathbf{1} = (1/\sqrt{nN})(1, \dots, 1)$, generates the moments of $X_{nN} = (1/\sqrt{nN}) \sum_{i,j} x_{i,j}$. We combine the procedure of Section 2.2 with the proof of Theorem 4.2 to evaluate this ratio. Combining (2.25) with (4.9) yields

$$\begin{aligned} K_n^{(N)} &= \frac{\beta^{nN/2}}{S_N^n} \int_{\mathbb{R}^{n \cdot N}} \prod_{i=1}^n d^N x_i \exp \left\{ \frac{-1}{2} (\mathbf{x}, (A - \kappa) \mathbf{x}) + z(\mathbf{1}, \mathbf{x}) \right\} \\ &= \frac{2^{n(N/2-1)} (\Gamma(N/2))^n}{N^{n(N-1)/2} \sqrt{\det(J - \kappa I)}} \exp \left\{ \frac{-z^2}{2\kappa} \right\} \\ &\leq \left(\sqrt{N} \int_0^\infty \frac{ds}{s} \exp \{ N g_n^{(N)}(s) \} \right)^n \end{aligned} \quad (4.13)$$

with

$$g_n^{(N)}(s) = \frac{\kappa}{2} s^2 + \ln s + f_n^{(N)}(s^2, s)$$

where the free energy

$$f_n^{(N)}(\beta, z) = \frac{1}{nN} \ln Z_n(\beta, z; \mathbf{1}, A)$$

is a smooth function of (β, z) such that $f_n^{(N)}(s^2, 0) = f_n^{(N)}(s^2)$ is given by (4.2). Analogously to (4.10), we have

$$\begin{aligned} K_n^{(N)} &\geq (NR_N^2)^{n/2} Z_n^{(N)}(1, \langle S(J - \kappa I)S \rangle \otimes I) \\ &= (NR_N^2)^{n/2} \exp \left\{ nN \left(\frac{1}{2} \kappa \langle s^2 \rangle + f_n^{(N)}(\langle s \rangle^2, \langle s \rangle) \right) \right\} \end{aligned} \quad (4.14)$$

with the constant a in normalization R_N chosen so that

$$\lim_{N \rightarrow \infty} \sqrt{\frac{N}{\beta}} R_N = \sqrt{\frac{2\pi}{-\left(g_n^{(\infty)}\right)''(\beta)}} \frac{1}{\beta}$$

and, recalling equation (2.15), the difference between the upper (4.13) and the lower bound (4.14) when the chemical potential κ is chosen as a function of β is $o(1)$.

One concludes from the last two equations (4.13) and (4.14) the following. If $\bar{s} = \bar{s}(\kappa)$ solves the equation (2.27), then

$$\left(\frac{\beta}{2} \kappa + \frac{1}{2} \ln \beta + f_n^{(N)}(\beta, z) \right) < -\frac{1}{2} (1 + \mathbb{E} \ln(J - \kappa)) - \frac{z^2}{2\kappa} + O(n) < \left(\frac{\kappa}{2} \bar{s}^2 + \ln \bar{s} + f_n^{(N)}(\bar{s}^2, \bar{s}) \right)$$

holds provided n and N sufficiently large and $\kappa = \kappa_{n,N}(\beta, z)$ solves

$$\beta = \mathbb{E} \frac{1}{-\Delta - \kappa} + \frac{z^2}{nN\kappa^2}$$

for some $O(n)$ constant independent of z . Substituting $\exp(nN f_n^{(N)}(\beta, z))$ in the numerator and denominator (with $z = 0$) of (4.11), gives

$$\Theta_n^{(N)}(\beta, z) = \exp \left(-nN \frac{\beta}{2} (\kappa - \mu) - \frac{nN}{2} \mathbb{E} \ln \frac{J - \kappa}{J - \mu} - \frac{z^2}{2\kappa} \right) \left(1 + o\left(\frac{1}{N}\right) \right)$$

and this implies

$$\ln \Theta(\beta, z) = \lim_{n, N \rightarrow \infty} \ln \Theta_n^{(N)}(\beta, z) = \frac{-z^2}{2\mu(\beta)}$$

for every $\beta < \beta_c$, uniformly in compacts of z .

□

Remark 4.4

1. In order to prove that the theorem holds for the hierarchical Laplacean we need only to prove that the functional defined by measure (1.1) satisfies reflection positivity.

4.3 Reflection Positivity for the Hierarchical Laplacean

The $O(N)$ Heisenberg hierarchical measure has been shown to satisfy reflection positivity by Watanabe who considered the model originally proposed by Dyson with $L^d = 2$. We extend his proof to the general case $L > 1$, $d \geq 1$.

Let $\theta = (\theta_1, \dots, \theta_K)$, $\theta_k \in \{0, 1\}$, be the binary representation of a point $i \in \{0, 1, \dots, 2^K - 1\}$. With respect to a reflection plane P_k at the k -th hierarchy the map r given by

$$(r\theta)_l = \begin{cases} \theta_l & \text{if } l \neq k \\ 1 - \theta_l & \text{if } l = k \end{cases}$$

acts as exchanging each pair of consecutive blocks of size 2^{k-1} indexed by $\tau = (\theta_{k+1}, \dots, \theta_K)$:⁴ $\{i_1, \dots, i_{2^{k-1}}, j_1, \dots, j_{2^{k-1}}\} \longrightarrow \{j_1, \dots, j_{2^{k-1}}, i_1, \dots, i_{2^{k-1}}\}$.

For the d -dimensional lattice, a point i is represented by $\theta = (\theta_1, \dots, \theta_K)$ where θ_k takes values in a box $B_L = \{0, 1, \dots, L-1\}^d$ with periodic boundary conditions: $\theta_k = (\theta_{k,1}, \dots, \theta_{k,d})$ with $\theta_{k,j} \bmod L \in \{0, 1, \dots, L-1\}$. As in the previous subsection, for each hierarchy k a reflection plane P is chosen perpendicular to coordinate axes cutting bonds, but not sites, of $\{0, 1, \dots, L-1\}^d$ dividing this box into two disjoint halves $B_L = B_L^+ \cup B_L^-$. With respect to a reflection plane P at k -th hierarchy the map r assigns to each θ its “reflected image” $r\theta$ with $(r\theta)_l = \theta_l$ if $l \neq k$ and $(r\theta)_k \in B_L^\pm$ if $\theta_k \in B_L^\mp$.

Let

$$\Lambda_\pm = \{\theta : \theta_k \in B_L^\pm\}$$

be the partition of $\Lambda_K = \{0, \dots, L^K - 1\}^d$ into two halves according to a plane P at hierarchy k and let \mathcal{P}_\pm denote the set of polynomials in x_θ , $\theta \in \Lambda_\pm$. The reflection map r induces a linear morphism $\pi_P : \mathcal{P}_+ \longrightarrow \mathcal{P}_-$, given by

$$\pi_P F(\{x_\theta\}_{\theta \in \Lambda_+}) = F(\{x_{r\theta}\}_{\theta \in \Lambda_+}) .$$

To prove reflection positivity of the $O(N)$ Heisenberg hierarchical measure, its enough to show that $-\Delta$ given by (3.2) is a reflection positivity interaction. For this, let $y = B^l x \in \mathbb{R}^{\Lambda_{K-l}}$ and notice that $y_\tau \in \mathcal{P}_\pm$ according to whether $\theta_k \in B_L^\pm$ are coordinate of $\tau = (\theta_{l+1}, \dots, \theta_K)$ for $l < k$. If $l \geq k$, then y can be decomposed as

$$y = y^+ + y^- , \quad y^\pm \in \mathcal{P}_\pm$$

with $y^- = \pi_P y^+$. We thus have

$$\begin{aligned} (y, y)_{\Lambda_{K-l}} &= \sum_{\tau: \theta_k \in B_L^+} |y_\tau|^2 + \sum_{\tau: \theta_k \in B_L^-} |y_\tau|^2 \\ &= \sum_{\tau: \theta_k \in B_L^+} |y_\tau|^2 + \sum_{\tau: \theta_k \in B_L^+} |\pi_P y_\tau|^2 \\ &= \sum_{\tau: \theta_k \in B_L^+} |y_\tau|^2 + \pi_P \sum_{\tau: \theta_k \in B_L^+} |y_\tau|^2 \equiv \|y\|_+^2 + \pi_P \|y\|_+^2 \end{aligned}$$

⁴Each pair of consecutive blocks would be reflected : $\{i_1, \dots, i_{2^{k-1}}, j_1, \dots, j_{2^{k-1}}\} \longrightarrow \{j_{2^{k-1}}, \dots, j_1, i_{2^{k-1}}, \dots, i_1\}$ if $(r\theta)_l = 1 - \theta_l$ holds for $l \leq k$. We use the exchange operation for simplicity.

by the morphism property, if $l < k$ and

$$\begin{aligned}
(y, y)_{\Lambda_{K-l}} &= \sum_{\tau} |y_{\tau}^{+} + y_{\tau}^{-}|^2 \\
&= \sum_{\tau} |y_{\tau}^{+}|^2 + \sum_{\tau} |y_{\tau}^{-}|^2 + 2 \sum_{\tau} y_{\tau}^{+} y_{\tau}^{-} \\
&\equiv \|y^{+}\|^2 + \pi_P \|y^{+}\|^2 + 2 \sum_{\tau} y_{\tau}^{+} \pi_P y_{\tau}^{+}
\end{aligned}$$

and these together with (3.2), according to [FILS], imply that $-\Delta$ is a reflection positivity interaction.

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